Closing Today: 2.1, 2.2, 2.3
Closing Tuesday: 2.5-6
Closing next Fri: 2.7, 2.7-8
Extended office hours today 1:30-3pm
in Com. B-006 (next to MSC)

Entry Task: **From HW**, evaluate:

1. $\lim_{t\to\pi/2}\left[\frac{\sin(t)+\sqrt{\sin^2(t)+3\cos^2(t)}}{2\cos^2(t)}\right] = +\infty$

2. $\lim_{t\to\pi/2}\left[\frac{\sin(t)-\sqrt{\sin^2(t)+3\cos^2(t)}}{2\sin^2(t)}\right] = 0$

3. $\lim_{t\to\pi/2}\left[\frac{\sin(t)-\sqrt{\sin^2(t)+3\cos^2(t)}}{2\cos^2(t)}\right]$

[1] $\frac{1+\sqrt{1+0}}{0} = \frac{2}{0}$

$\Rightarrow +\infty$, $-\infty$, or DNE

Since the numerator goes to +2 and the denominator goes to 0 through positive numbers we get $\boxed{+\infty}$

[2] $\frac{1-\sqrt{1+0}}{2} = \boxed{0}$ DONE!

[3] $\frac{1-\sqrt{1+0}}{0} = 0/0 \leftarrow$ ALGEBRA!

MULTIPLY TOP/BOT BY
$\sin(t) + \sqrt{\sin^2(t)+3\cos^2(t)}$

$$= \lim_{t\to\pi/2} \frac{\sin^2(t) - (\sin^2(t) + 3\cos^2(t))}{2\cos^2(t)\left(\sin(t) + \sqrt{\sin^2(t)+3\cos^2(t)}\right)}$$

$$= \lim_{t\to\pi/2} \frac{-3}{2\left(\sin(t) + \sqrt{\sin^2(t)+3\cos^2(t)}\right)} = \frac{-3}{2(1+\sqrt{1+0})} = \boxed{-\frac{3}{4}}$$

2.5 Continuity (continued…)

A function, $f(x)$, is **continuous at x = a** if

$$\lim_{x \to a} f(x) = f(a)$$

i.e. the following must be equal:

(i) $\lim_{x \to a^-} f(x)$

(ii) $\lim_{x \to a^+} f(x)$

(iii) $f(a)$

Example: Find the value of *c* that makes the function continuous everywhere:

$$f(x) = \begin{cases} \dfrac{(x+1)^2 - 16}{x-3} & \text{, if } x < 3; \\ 2x^2 + c & \text{, if } x \geq 3. \end{cases}$$

$$\lim_{x \to 3^-} \frac{(x+1)^2 - 16}{x-3}$$

$$= \lim_{x \to 3^-} \frac{x^2 + 2x + 1 - 16}{x-3} \qquad (x^2 + 2x - 15$$

$$= \lim_{x \to 3^-} \frac{(x-3)(x+5)}{x-3}$$

$$= 8$$

$$\lim_{x \to 3^+} 2x^2 + c = 2(3)^2 + c$$

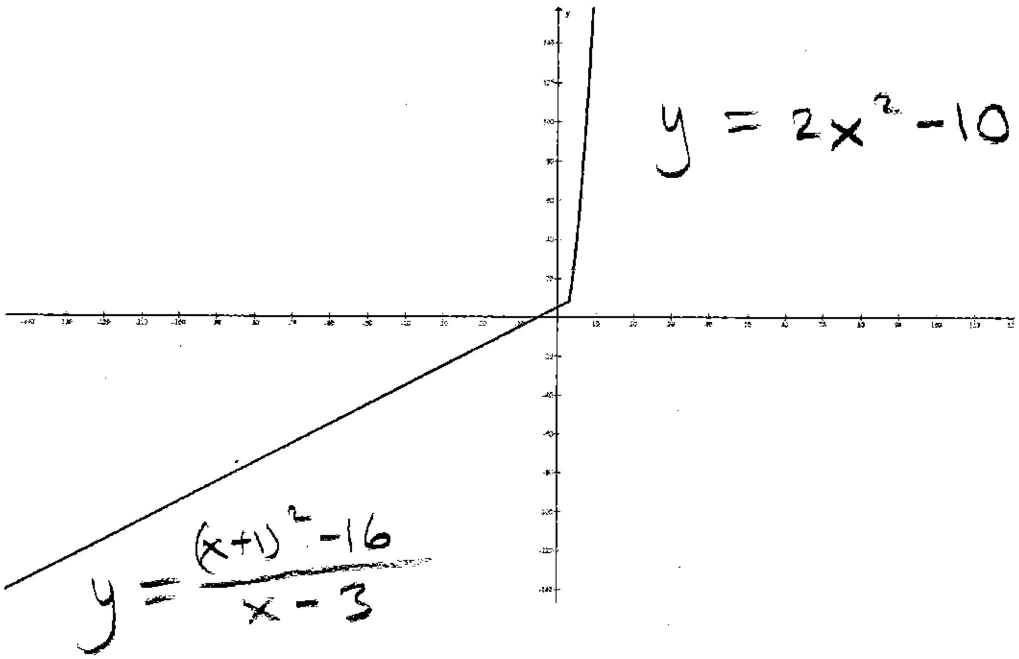
$$= 18 + c = f(3)$$

WANT

$$18 + c = 8$$

$$\boxed{c = -10}$$

f(x)



Example:

$$h(x) = \begin{cases} ax^2 + 6 & ,\text{if } x < 1; \\ b & ,\text{if } x = 1; \\ \dfrac{x+49}{x+a} & ,\text{if } x > 1. \end{cases}$$

Find the values of *a* and *b* that will make $h(x)$ continuous *everywhere*.

WE ARE "WORRIED" ABOUT $x=1$ AND $x=-a$.

$\lim_{x\to 1^-} h(x) = a+6$

$\lim_{x\to 1^+} h(x) = \frac{50}{1+a}$

$h(1) = b$

WANT THE SAME!

$a+6 = \frac{50}{1+a}$

$\Rightarrow (a+6)(1+a) = 50$

$a^2+7a+6 = 50$

$a^2+7a-44 = 0$

$(a+11)(a-4) = 0$

$\Rightarrow a=-11$ or $a=4$

$\Downarrow$ $b = a+6 = -5$

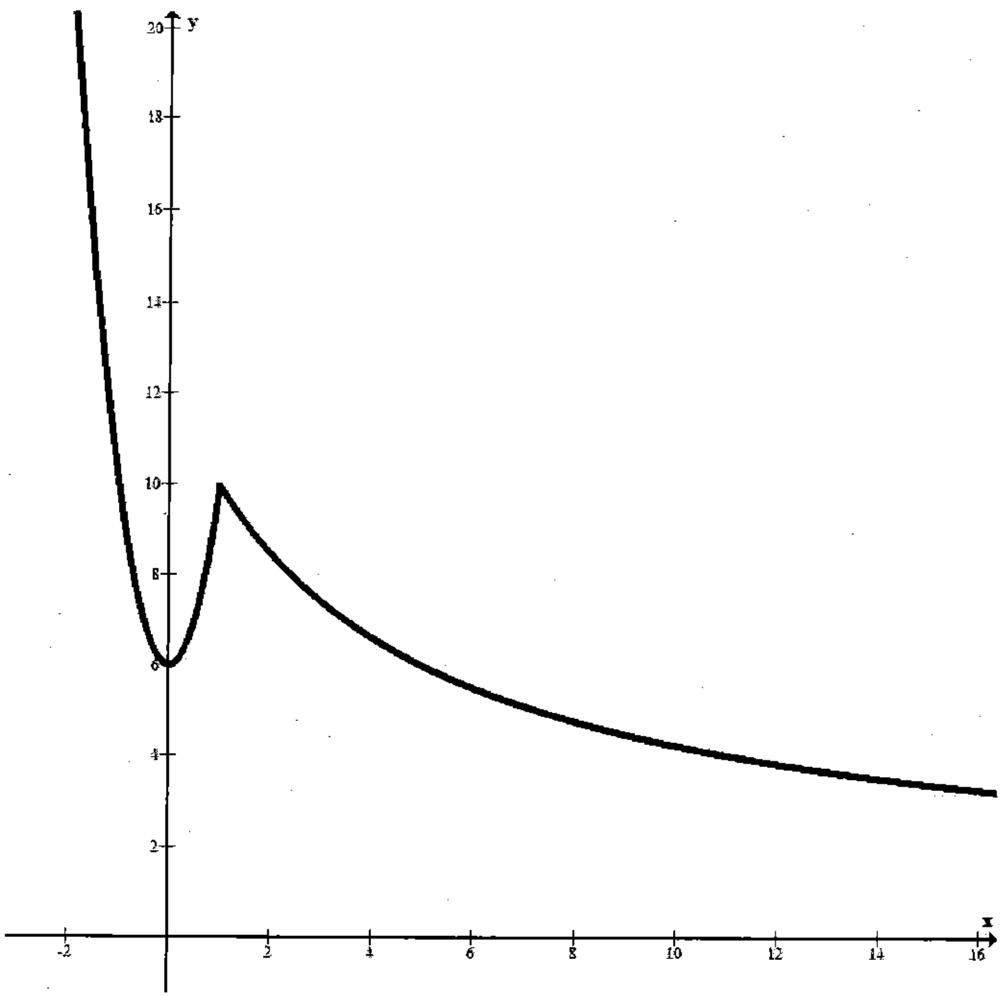
$\Downarrow$ $b = a+6 = 10$

$a=-11$ can't be the answer because then $\frac{x+49}{x-11}$ would not be defined at $x=11$, BUT THIS IS NOT A PROBLEM FOR $\frac{x+49}{x+4}$ BECAUSE -4 IS NOT IN THAT DOMAIN

$a=4$

$b=10$

h(x)



For 8 more continuity problems like these, see my online practice sheet: "Continuity Practice Problems" (There are posted solutions as well).

Continuity Practice

Some students say they have trouble with multipart functions. Other say they have issues with continuity problems. Here is a random assortment of old midterm questions that pertain to continuity and multipart functions. See if you can complete these problems. Solutions are posted online.

Remember a function $f(x)$ is *continuous* at $x = a$ if $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^-} f(x)$, $f(a)$ are all defined and are all the same.

1. Let $f(x) = \begin{cases} \cos(x) + 1 & \text{, if } x \leq 0; \\ 2 - 3x & \text{, if } x > 0. \end{cases}$ Determine if this function is continuous at $x = 0$.

2. Let $f(x) = \begin{cases} \frac{\sqrt{9x^4+x^2}}{5x^2+3x+1} & \text{, if } x \leq 0; \\ x & \text{, if } x < 0. \end{cases}$. Is f continuous at $x = 0$?

3. Let $f(x) = \begin{cases} e^x & \text{, if } x < 0; \\ 9x^2 + x + 1 & \text{, if } x \geq 0. \end{cases}$. Is f continuous at $x = 0$?

4. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 3 & \text{, if } x \neq 0; \\ 1 & \text{, if } x = 0. \end{cases}$. Is f continuous at $x = 0$?

5. Let $f(x) = \begin{cases} -x + c & , \text{if } x \leq 1; \\ 6 - 2x^2 & , \text{if } x > 1. \end{cases}$ Find a value of c so that $f(x)$ is continuous at $x = 1$.

6. Let $f(x) = \begin{cases} \frac{x^2-9}{x-3} & , \text{if } x < 3; \\ cx^2 + 10 & , \text{if } x \geq 3. \end{cases}$ Find the value of c so that $f(x)$ is continuous at $x = 3$.

7. Let $G(x) = \begin{cases} \frac{1}{(x+3)^2} & , \text{if } x \leq -1; \\ 2 - x & , \text{if } -1 < x \leq 1; \\ \frac{3}{x+2} & , \text{if } x > 1. \end{cases}$. Find all values of x where G is not continuous.

Solutions:

1. Let $f(x) = \begin{cases} \cos(x)+1 & , \text{if } x \leq 0; \\ 2-3x & , \text{if } x > 0. \end{cases}$ Determine if this function is continuous at $x = 0$.

Solution:
1. The function is defined at $x = 0$ and the value is $f(0) = \cos(0) + 1 = 2$.
2. Since $y = \cos(x) + 1$ is continuous at $x = 0$, we have:
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x) + 1 = \cos(0) + 1 = 2.$$
3. Since $y = 2 - 3x$ is continuous at $x = 0$, we have:
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 - 3x = 2 - 3(0) = 2.$$
Since all three of these values are the same, the function is continuous at $x = 0$.

2. Let $f(x) = \begin{cases} \frac{\sqrt{9x^4+x^2}}{5x^2+3x+1} & , \text{if } x \leq 0; \\ x & , \text{if } x > 0. \end{cases}$. Is f continuous at $x = 0$?

Solution:
1. The function is defined at $x = 0$ and its value is $f(0) = \frac{\sqrt{9(0)^4+(0)^2}}{5(0)^2+3(0)+1} = 0$.
2. Since $y = \frac{\sqrt{9x^4+x^2}}{5x^2+3x+1}$ is continuous at $x = 0$, we have:
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sqrt{9x^4+x^2}}{5x^2+3x+1} = 0.$$
3. Since $y = x$ is continuous at $x = 0$, we have:
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.$$
Since all three of these values are the same, the function is continuous at $x = 0$.

3. Let $f(x) = \begin{cases} e^x & , \text{if } x < 0; \\ 9x^2+x+1 & , \text{if } x \geq 0. \end{cases}$. Is f continuous at $x = 0$?

Solution:
1. The function is defined at $x = 0$ and its value is $f(0) = 9(0)^2 + (0) + 1 = 1$.
2. Since $y = e^x$ is continuous at $x = 0$, we have:
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1.$$
3. Since $y = x$ is continuous at $x = 0$, we have:
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 9x^2 + x + 1 = 1.$$
Since all three of these values are the same, the function is continuous at $x = 0$.

4. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 3 & , \text{if } x \neq 0; \\ 1 & , \text{if } x = 0. \end{cases}$. Is f continuous at $x = 0$?

Solution:
1. The function is defined at $x = 0$ and its value is $f(0) = 1$.
2. Now we use the squeeze theorem to find the value of the limit.
Since $-1 \leq \sin\left(t\frac{1}{x}\right) \leq 1$ for all values of x, we can multiply by x^2 to get $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all values of x. Since $\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$, we conclude that the function between them also approaches zero. Therefore $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$, which implies $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) + 3 = 3$.

Since the value of limit does NOT equal the value of the function, $f(x)$ is NOT continuous at $x = 0$.

5. Let $f(x) = \begin{cases} -x+c & , \text{if } x \leq 1; \\ 6-2x^2 & , \text{if } x > 1. \end{cases}$ Find a value of c so that $f(x)$ is continuous at $x = 1$.

Solution:

1. The function is defined at $x = 1$ and its value is $f(1) = -1 + c$.
2. Since $y = -x + c$ is continuous at $x = 1$, we have:
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -x + c = -1 + c.$$
3. Since $y = 6 - 2x^2$ is continuous at $x = 1$, we have:
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 6 - 2x^2 = 6 - 2(1)^2 = 4.$$

In order to make all three of these the same, we need $-1 + c = 4$. Thus, $c = 5$.

6. Let $f(x) = \begin{cases} \frac{x^2-9}{x-3} & , \text{if } x < 3; \\ cx^2+10 & , \text{if } x \geq 3. \end{cases}$ Find the value of c so that $f(x)$ is continuous at $x = 3$.

Solution:

1. The function is defined at $x = 3$ and its value is $f(3) = c(3)^2 + 10 = 9c + 10$.
2. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x^2-9}{x-3} = \lim_{x\to 3^-} \dfrac{(x-3)(x+3)}{x-3} = 6.$
3. Since $y = cx^2 + 10$ is continuous at $x = 3$, we have:
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} cx^2 + 10 = 9c + 10.$$

In order to make all three of these the same, we need $9c + 10 = 6$. Thus, $c = -\frac{4}{9}$.

7. Let $G(x) = \begin{cases} \frac{1}{(x+3)^2} & , \text{if } x \leq -1; \\ 2-x & , \text{if } -1 < x \leq 1; \\ \frac{3}{x+2} & , \text{if } x > 1. \end{cases}$. Find all values of x where G is not continuous.

Solution: There are four points to immediately consider: $x = -3$ and $x = -2$ because they make a denominator zero as well as $x = -1$ and $x = 1$ because the function rule changes at these values.

$\mathbf{x = -3}$: Since $y = \frac{1}{(x+3)^2}$ is discontinuous at $x = -3$ and $G(x)$ uses this rule for $x < -1$, we see that $G(x)$ is NOT continuous at $x = -3$.

$\mathbf{x = -2}$: Even through $y = \frac{3}{x+2}$ is discontinuous at $x = -2$, the function $G(x)$ only uses the rule $y = \frac{3}{x+2}$ for values where $x > 1$ and the rule it does use at $x = -2$ is continuous at that value. So $G(x)$ is continuous at $x = -2$.

$\mathbf{x = -1}$: $\lim_{x\to -1^-} G(x) = \dfrac{1}{(-1+3)^2} = \dfrac{1}{4}$ and $\lim_{x\to -1^+} G(x) = 2 - (-1) = 3$. Since these are not the same, the function $G(x)$ is NOT continuous at $x = -1$.

$\mathbf{x = 1}$: $\lim_{x\to 1^-} G(x) = 2 - (1) = 1$ and $\lim_{x\to 1^+} G(x) = \dfrac{3}{1+3} = 1$. Since these ARE the same and they equal the value of the function at $x = 1$, the function $G(x)$ is continuous at $x = 1$.

Therefore, the function $G(x)$ is continuous everywhere except $x = -3$ and $x = -1$.

Theorem:

If $f(x)$ is continuous at $x = b$, and

$$\lim_{x \to a} g(x) = b$$

then

$$\lim_{x \to a} f\big(g(x)\big) = f(b).$$

Example:

Find

$$\lim_{x \to 9} \ln\left(\frac{\sqrt{x} - 3}{x - 9}\right) = \boxed{\ln\left(\tfrac{1}{6}\right)} = \ln(1) - \ln(6) = -\ln(6)$$

$$= \ln\left(\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}\right)$$

$$\lim_{x \to 9} \frac{(\sqrt{x} - 3)}{(x - 9)} \frac{(\sqrt{x} + 3)}{(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \frac{1}{\sqrt{9} + 3} = \boxed{\tfrac{1}{6}}$$

2.6 Limits "at" Infinity
(*Horizontal Asymptotes*)

Goal: Study "long term" behavior.

$$\lim_{x\to\infty} f(x) = L$$

"the limit of $f(x)$, as x goes to infinity is L",
as x takes on larger and larger positive numbers,
$y = f(x)$ takes on values closer and closer to L.

Similarly,

$$\lim_{x\to-\infty} f(x) = L$$

"the limit of $f(x)$, as x goes to negative infinity is L".

Important limits to know:

1. For any <u>positive</u> number *n*,

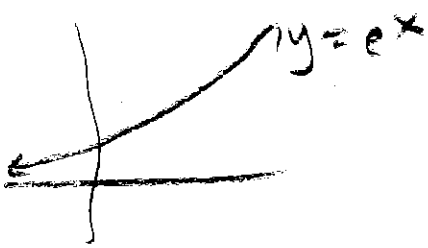
$$\lim_{x\to\infty} x^{-n} = \lim_{x\to\infty} \frac{1}{x^n} = 0.$$

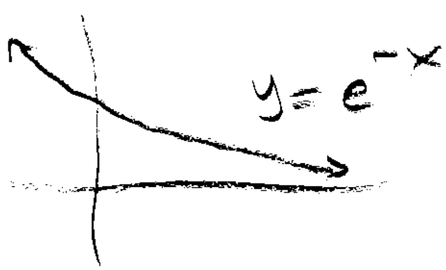
$$\lim_{x\to-\infty} x^{-n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

(if defined)

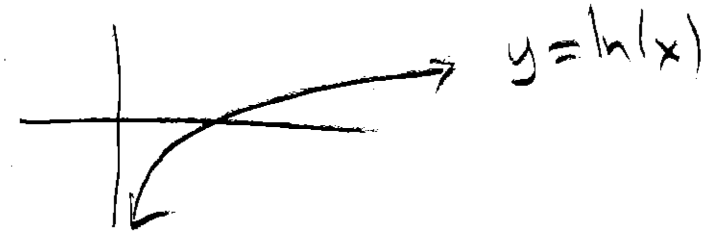
2. $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0.$

$\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to-\infty} e^{-x} = \infty$



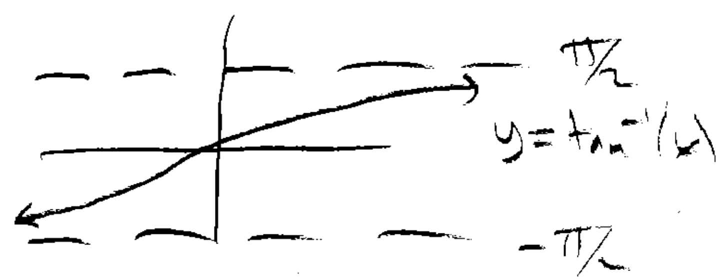


3. $\lim_{x\to\infty} \ln(x) = \infty$



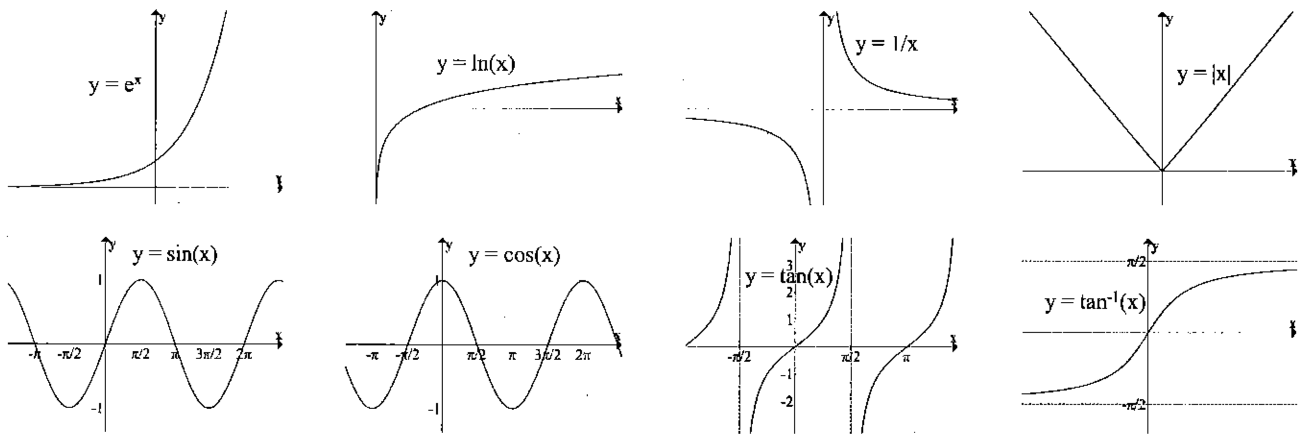
4. $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2},$

$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$



Functions and Limits Review

In order to do well in this course and before we can understand limits, we must know our basic functions. Here is a quick visual review of graphs of some functions that students sometimes forget:



On the exams, you are allowed to use what you see in these graphs. For example, by looking at the graphs you immediately know all of the following:

$\lim_{x\to-\infty} e^x = 0$	$\lim_{x\to\infty} e^x = \infty$	$\lim_{x\to0+} \ln(x) = -\infty$	$\lim_{x\to\infty} \ln(x) = \infty$
$\lim_{x\to-\infty} \frac{1}{x} = 0$	$\lim_{x\to0^-} \frac{1}{x} = -\infty$	$\lim_{x\to0+} \frac{1}{x} = \infty$	$\lim_{x\to\infty} \frac{1}{x} = 0$
$\lim_{x\to\frac{\pi}{2}^-} \tan(x) = \infty$	$\lim_{x\to-\frac{\pi}{2}^+} \tan(x) = -\infty$	$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$	$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

One Special Note:

We will make use of the particular fact $\lim_{x\to0} \frac{\sin(x)}{x} = 1$ (if x is in radians!).

A proof of this fact is posted on my course website and is in the book. The variable x is not important, what this says is that $\lim_{BLAH\to0} \frac{\sin(BLAH)}{BLAH} = 1$. So for example: $\lim_{x\to0} \frac{\sin(10x)}{10x} = 1$ and $\lim_{x\to0} \frac{\sin(31x)}{31x} = 1$.

Now if the denominator does not match the numerator, then we can do a bit of rearranging of fractions to make them match.

For example: $\lim_{x\to0} \frac{\sin(5x)}{x} = \lim_{x\to0} \frac{\sin(5x)}{x}\frac{5}{5} = \lim_{x\to0} 5\frac{\sin(5x)}{5x} = 5\cdot1 = 5.$

Practice, practice, practice!:

Using the facts above along with the limit strategies discussed in class and summarized in my other review sheets, go practice your limit methods. There is a compilation of old final problems posted online, check them out. Also look at old midterm exams. Besides the departmental old exam archive, I also maintain my own archive of many old exams by me and other instructors I know, check out many, many, many old exams. You need to expose yourself to lots of different problems!

Strategies to compute $\lim_{x\to\infty} f(x)$

1. Is it a standard one from my list on the last page?
   If so, *done*. If not, go to next step.

2. Combine into one fraction.
3. **Use algebra to *rewrite* in terms of known limits from previous page**:

Strategy 1: Multiply top/bot by $\frac{1}{x^a}$, where *a* is the largest power.

Strategy 2: Multiply top/bot by $\frac{1}{e^{rx}}$.

Strategy 3: Multiply by conjugate.

Strategy 4: Combine into one fraction.

Examples:

1. $\lim_{x\to\infty} \frac{(3+x^4-x)}{(4x^2+1-6x^4)} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$

$= \lim_{x\to\infty} \frac{\frac{3}{x^4} + 1 - \frac{1}{x^3}}{\frac{4}{x^2} + \frac{1}{x^4} - 6}$

$= \frac{0+1-0}{0+0-6} = \boxed{-\frac{1}{6}}$

could also do directly, just wanted to illustrate combining

2. $\lim_{x\to\infty} \left(\frac{x}{x+2} - \frac{1}{x}\right)$

$= \lim_{x\to\infty} \frac{x^2}{x(x+2)} - \frac{(x+2)}{x(x+2)}$

$= \lim_{x\to\infty} \frac{x^2-x-2}{x(x+2)}$

$= \lim_{x\to\infty} \frac{(x^2-x-2)}{(x^2+2x)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{1-\frac{1}{x}-\frac{2}{x^2}}{1+\frac{2}{x}} = \frac{1-0-0}{1+0} = \boxed{1}$

3. $\lim_{x\to\infty} \frac{3+5e^{(2x)}}{2e^{x}+4e^{(2x)}}$ $\quad \frac{\frac{1}{e^{(2x)}}}{\frac{1}{e^{(2x)}}}$

$= \lim_{x\to\infty} \frac{\frac{3}{e^{2x}} + 5}{\frac{2}{e^{x}} + 4}$

$= \frac{0+5}{0+4} = \boxed{5/4}$

5. $\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x + 1}}{2x^3 - x^2}$ $\quad \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$x^3 = -\sqrt{x^6}$

$\lim_{x\to-\infty} \frac{-\sqrt{9 - 1/x^5 + 1/x^6}}{2 - 1/x}$

$= \frac{-\sqrt{9-0+0}}{2-0} = \boxed{-3/2}$

$\frac{\sqrt{9x^6 - x + 1}}{-\sqrt{x^6}}$

Note: $\sqrt{x^2} = x$, if $x \geq 0$, and
$\sqrt{x^2} = -x$, if $x < 0$.

4. $\lim_{x\to\infty} \frac{\sqrt{9x^6 - x + 1}}{2x^3 - x^2}$ $\quad \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$ $\leftarrow \frac{1}{\sqrt{x^6}}$

$= \lim_{x\to\infty} \frac{\sqrt{9 - 1/x^5 + 1/x^6}}{2 - 1/x}$

$= \frac{\sqrt{9-0+0}}{2-0} = \boxed{3/2}$

$\frac{\sqrt{9x^6 - x + 1}}{\sqrt{x^6}}$

$\sqrt{\frac{9x^6 - x + 1}{x^6}}$

5. $\lim_{x\to\infty} \left(\sqrt{3+2x+x^2} - x\right)\dfrac{\left(\sqrt{3+2x+x^2}+x\right)}{\left(\sqrt{3+2x+x^2}+x\right)}$

$$= \lim_{x\to\infty} \frac{(3+2x+x^2) - x^2}{\left(\sqrt{3+2x+x^2}+x\right)} \quad \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{3/x + 2}{\frac{\sqrt{3+2x+x^2}}{x} + 1}$$

$\leftarrow \sqrt{x^2}$

$$= \lim_{x\to\infty} \frac{3/x + 2}{\sqrt{\frac{3}{x^2}+\frac{2}{x}+1} + 1}$$

$$= \frac{0+2}{\sqrt{0+0+1}+1} = \frac{2}{2} = \boxed{1}$$

Strategies to compute: $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]$

Special note: If given two fractions, combine them (common denom).

Try plugging in the value:

1. **If denominator ≠ 0, done!**
2. **If denom = 0 & numerator ≠ 0,** the answer is -∞, +∞ or DNE. Examine the sign of the output from each side.
3. **If denom = 0 & numerator = 0,** Use algebra to simplify and cancel until either the numerator or denominator is not zero.

Strategy 1: Factor/Cancel
Strategy 2: Simplify Fractions
Strategy 3: Expand/Simplify
Strategy 4: Multiply by Conjugate
(if you see radicals)

Strategies to compute: $\lim_{x\to\infty} f(x)$

Special note: Combine into one fraction (might need conjugate if given two terms involving a radical).

1. Is it a known limit?

$$\lim_{x\to\infty}\frac{1}{x^a}=0, \text{if } a>0;\ \lim_{x\to\infty} e^{-x}=0;$$

$$\lim_{x\to\infty}\ln(x)=\infty;\ \lim_{x\to\infty}\tan^{-1}(x)=\frac{\pi}{2}.$$

2. Rewrite in terms of known limits:

Strategy 1: Multiply top/bottom by $\frac{1}{x^a}$,
where *a* is the largest power.
Strategy 2: Multiply top/bottom by e^{-rx}.

Special note:
If x is positive, then $x = \sqrt{x^2}$.
If x is negative, then $x = -\sqrt{x^2}$.